

ON A CONJECTURE OF SERRE ON ABELIAN THREEFOLDS

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En genre 3, le théorème de Torelli s'applique de façon moins satisfaisante : on doit extraire une mystérieuse racine carrée (J.-P.S., Collected Papers, n° 129)

ABSTRACT. In this article, we give a reformulation of a result from Howe, Leprevost and Poonen on a three dimensional family of abelian threefolds. We also link their result to a conjecture of Serre on a precise form of Torelli theorem for genus 3 curves.

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1. INTRODUCTION

1.1. Geometric Torelli's theorem. Let K be an algebraically closed field. If X is a (smooth algebraic projective) curve of genus g over K , the Jacobian $\text{Jac } X$ of X is an abelian variety of dimension g , and $\text{Jac } X$ has a canonical principal polarization λ . We obtain in this way a morphism

$$\text{Jac} : \mathbf{M}_g \longrightarrow \mathbf{A}_g$$

from the space \mathbf{M}_g of (K -isomorphism classes of) curves of genus g to the space \mathbf{A}_g of (K -isomorphism classes of) g -dimensional principally polarized abelian varieties (p.p.a.v.).

According to Torelli's Theorem, proved one century ago, the map $X \mapsto (\text{Jac } X, \Theta)$ is injective. An algebraic proof was provided by Weil [18] half a century ago, and it is a long time studied question to characterize the image of this map.

If $g = 3$, these spaces are both of dimension $3g - 3 = g(g + 1)/2 = 6$. According to Hoyt [7] and Oort and Ueno [15], the image of \mathbf{M}_g is exactly the space of indecomposable principally polarized threefolds. Recall that (A, λ) is decomposable if there is an abelian subvariety B of A neither equal to 0 nor to A , such that the restriction of λ to B is a principal polarization, and indecomposable otherwise. This was a problem left unsolved by Weil in [18].

Given a principally polarized abelian threefold (A, λ) over K , two natural questions arise :

- (i) How can we decide if the polarization is indecomposable ?
- (ii) How can we decide if A is the Jacobian of a hyperelliptic curve ?

Actually, both questions were answered by Igusa in 1967 [9] when $K = \mathbb{C}$, making use of a particular modular form χ_{18} on the Siegel upper half-space (see Th. 3.5.2 below).

1.2. Arithmetic Torelli's theorem. Assume now that K is an arbitrary field. Then, as Serre noticed in [12], the above correspondence is no longer true. Let (A, λ) be a p.p.a.v. of dimension g over K , and assume that (A, λ) is isomorphic over \overline{K} to the Jacobian of a curve \mathcal{X} of genus g .

Theorem 1.2.1 (Serre). *The following alternative holds :*

- (i) *If \mathcal{X} is hyperelliptic, there exists a model X/K of \mathcal{X} and a K -isomorphism between the p.p.a.v. $(\text{Jac } X, \Theta)$ and (A, λ) .*
- (ii) *If \mathcal{X} is non hyperelliptic, there exists a model X/K of \mathcal{X} and a quadratic character $\varepsilon : \text{Gal}(K_s/K) \rightarrow \{\pm 1\}$ such that $(\text{Jac } X, \Theta)$ is isomorphic to the twist $(A, \lambda)_\varepsilon$ of (A, λ) by ε .*

In particular, if ε is not trivial, this implies that $\text{Jac } X$ is not isomorphic to A over K , but only over a quadratic extension, and (A, λ) is not isomorphic over K to the Jacobian of a curve. \square

1.3. Serre's conjecture. Let us come back to the case $g = 3$. Let there be given an indecomposable principally polarized abelian threefold (A, λ) defined over K . In a letter to Top [17] in 2003, J.-P. Serre asked two questions:

- (i) How to decide, knowing only (A, λ) , that X is hyperelliptic ?
- (ii) If X is not hyperelliptic, how to find the quadratic character ε ?

He proposed, in the case $K \subset \mathbb{C}$, the following conjecture :

Conjecture 1.3.1. *Let (A, λ) be an indecomposable principally polarized abelian threefold over K isomorphic over \overline{K} to the Jacobian of a curve \mathcal{X} of genus 3. Then there is an invariant $\chi_{18}(A, \lambda)$ such that*

- (i) $\chi_{18}(A, \lambda) = 0$ *is and only if \mathcal{X} is hyperelliptic;*

- (ii) *the character ε is the one defined by the action of $\text{Gal}(\bar{K}/K)$ on the square root of $\chi_{18}(A, \lambda)$.*

We use here the notation χ_{18} to emphasize that this conjecture was inspired by the results obtained by Igusa, and also much earlier by Klein [11] (see the remark after Cor. 4.3.3). In this article Klein relates (up to an undetermined constant) the modular form χ_{18} and the square of the discriminant of the quartic \mathcal{X} (when $\chi_{18}(\tau) \neq 0$). This invariant seemed to Serre a good choice to find this “mysterious square root”.

We plan to answer in the affirmative this conjecture for a family of abelian threefolds which are isogenous to the product of three elliptic curves (see Cor. 4.3.2). This will rely on the work of Howe, Leprevost and Poonen [8] for which we propose a natural rephrasing. For any field K of characteristic different from 2, they consider abelian threefolds (A, λ) defined as a quotient of three elliptic curves (with the trivial polarization) by a certain subgroup of 2-torsion points. For this three-dimensional family, they make explicit the equation of the related curve and express the character ε by a invariant \mathbf{T} involving the coefficients of the elliptic curves. In the first part, we show that \mathbf{T} can be naturally interpreted as a determinant. In a second phase, we take $K \subset \mathbb{C}$ and by uniformization, we express \mathbf{T} in terms of certain Thetanullwerte of the elliptic curves. Then using the duplication and transformation formula we express the modular form $\chi_{18}(A, \lambda)$ in terms of the same Thetanullwerte and compare the two expressions. We also obtain a proof of Klein’s result in this particular case and give the constant involved (see Cor. 4.3.3).

We describe now briefly the different sections. In Sec. 2, we define Ciani quartics, go back to the aforementioned results of [8], and show the relation with Serre’s conjecture (§ 2.5). In Sec. 3, we recall some general facts about abelian varieties over \mathbb{C} (of arbitrary dimension) and introduce the modular function χ_k (§ 3.5). We prove Serre’s conjecture in Sec. 4. Finally, an appendix gathers some technical proofs, in particular the modularity of the form χ_k .

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2. CIANI QUARTICS

In this section we reformulate a result of [8] on a three-dimensional family of non-hyperelliptic genus 3 curves. In particular, this gives a more natural point of view on Prop. 15 of [8].

2.1. Definition of Ciani quartics. Edgardo Ciani gave in 1899 [3] a classification of nonsingular complex plane quartics curves based on the number of involutions in their automorphism group. We describe below the family of quartics admitting (at least) two commuting involutions (different from identity).

Let K be a field with $\text{char } K \neq 2$, and $\mathbf{Sym}_3(K)$ the vector space of symmetric matrices of size 3 with coefficients in K . Let

$$Q_m(x, y, z) = {}^t v.m.v, \quad v = (x^2, y^2, z^2), \quad m = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \in \mathbf{Sym}_3(K).$$

Then

$$Q_m(x, y, z) = a_1 x^4 + a_2 y^4 + a_3 z^4 + 2(b_1 y^2 z^2 + b_2 x^2 z^2 + b_3 x^2 y^2)$$

is a ternary quartic, and the map $m \mapsto Q_m$ is an isomorphism of $\mathbf{Sym}_3(K)$ to the vector space of ternary quartic forms invariant under the three involutions

$$\sigma_1(x, y, z) = (-x, y, z), \quad \sigma_2(x, y, z) = (x, -y, z), \quad \sigma_3(x, y, z) = (x, y, -z).$$

The form Q_m is the zero locus of a plane quartic curve X_m , whose automorphism group contains the Vierergruppe $V_4 = (\mathbb{Z}/2\mathbb{Z})^2$.

If X_m is a nonsingular curve, we say that X_m is a *Ciani quartic* and that Q_m is a *Ciani form*. Now, E. Ciani (*loc. cit.*) proved that a plane quartic admitting two commuting involutions is geometrically isomorphic to a Ciani quartic (a more recent reference is [1]).

Proposition 2.1.1. *If X is a plane quartic curve defined over K , admitting at least two commuting involutions, also defined over K , then there is $m \in \mathbf{Sym}_3(K)$ such that X is isomorphic to X_m over K .*

Proof. Let M_1 and M_2 in $\mathrm{PGL}_3(K)$ inducing two commuting involutions of X . Then $M_i^2 = \alpha_i \mathbf{I}$ with $\alpha_i \in K$ and $\det(M_i)^2 = \alpha_i^3$, hence α_i is a square and we can assume, by dividing M_i by $\sqrt{\alpha_i}$, that $M_i^2 = \mathbf{I}$. The two matrices M_i commute so we can diagonalize them in the same basis : after a change of coordinates, we can suppose that M_i are (projectively) equal to

$$I_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This implies that a quartic equation $Q(x, y, z) = 0$ of X in these new coordinates must be invariant by the involutions σ_1 and σ_2 above, hence, Q is a Ciani form. \square

2.2. Discriminant of a ternary form. Our definition of a Ciani form includes that its zero locus must be a nonsingular curve. This condition is fulfilled if and only if the discriminant of the form is not 0. In order to obtain a criterium for this condition, we develop an algorithm for the discriminant of a general ternary form. The *multivariate resultant* $\mathrm{Res}(f_1, \dots, f_n)$ of n forms f_1, \dots, f_n in n variables with coefficients in a field K is an irreducible polynomial in the coefficients of f_1, \dots, f_n which vanishes whenever f_1, \dots, f_n have a common non-zero root. One requires that the resultant is irreducible over \mathbb{Z} , i.e. it has integral coefficients with greatest divisor equal to 1, and moreover

$$\mathrm{Res}(x_1^{d_1}, \dots, x_n^{d_n}) = 1$$

for any $(d_1, \dots, d_n) \in \mathbb{N}^n$. The resultant exists and is unique. There is a remarkable determinantal formula for the resultant of 3 ternary forms of the same degree d , due to Sylvester; see [6] for a modern exposition and a proof. We give this formula in the case $d = 3$. Then $\mathrm{Res}(f_1, f_2, f_3)$ is a form of degree 27 in 30 unknowns. We shall express $\mathrm{Res}(f_1, f_2, f_3)$ as the determinant of a square matrix of size 15.

Let I_d be the set of sequences $\nu = (\nu_1, \nu_2, \nu_3)$ with $\nu_1 + \nu_2 + \nu_3 = d$. The monomials $x^\nu = x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3}$, where $\nu \in I_d$, form the canonical basis of the space V_d of ternary forms of degree d , which is of dimension $(d+1)(d+2)/2$.

For any monomial $x^\nu \in V_2$ with $\nu_1 + \nu_2 + \nu_3 = 2$, we choose arbitrary representations

$$f_i = x_1^{\nu_1+1} f_{i,1} + x_2^{\nu_2+1} f_{i,2} + x_3^{\nu_3} f_{i,3} \quad (1 \leq i \leq 3),$$

where $f_{i,j}$ are forms of degree $2 - \nu_j$, for $1 \leq i, j \leq 3$. Such a representation is always possible, although not unique. Now we define

$$S(x^\nu) = \det \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}.$$

Note that this determinant is indeed a ternary form of degree

$$(2 - \nu_1) + (2 - \nu_2) + (2 - \nu_3) = 6 - 2 = 4.$$

Since the monomials x^ν with $\nu \in I_2$ make up a basis of V_2 , we have thus defined a linear map $S : V_2 \longrightarrow V_4$. We consider the linear map

$$T : V_1 \times V_1 \times V_1 \times V_2 \longrightarrow V_4$$

given by

$$T(l_1, l_2, l_3, g) = l_1 f_1 + l_2 f_2 + l_3 f_3 + S(g).$$

Now one proves that the determinant of T is independent of the choices made in the definition of S , and *Sylvester's formula* holds :

$$\text{Res}(f_1, f_2, f_3) = \det T.$$

Generally speaking, the matrix of T involves 864 monomials. Now, let Q be a ternary form of degree d , and X be the plane projective curve which is the zero locus of Q . Call q_1, q_2, q_3 the partial derivatives of Q . The *discriminant* of Q is $\text{Disc } Q = \text{Res}(q_1, q_2, q_3)$. It is a form of degree $3(d-1)^2$ in the coefficients of Q , and X is non singular if and only if $\text{Disc } Q \neq 0$. The discriminant is an invariant of ternary forms : if $g \in \text{GL}_3(K)$, then

$$(1) \quad \text{Disc}(Q \circ g) = (\det g)^w \text{Disc } Q, \quad \text{where } w = d(d-1)^2.$$

If Q is a quartic form $\text{Disc } Q$ is a form of degree 27 in the coefficients, with $w = 36$ in (1). Applying Sylvester's formula, we get :

Proposition 2.2.1. *Let $m \in \mathbf{Sym}_3(K)$ and $c_i = a_j a_k - b_i^2$ is the cofactor of a_i for $1 \leq i \leq 3$. If (q_1, q_2, q_3) are the partial derivatives of the ternary quartic Q_m , then $\text{Disc } Q_m = 2^{54} \text{D}(m)$, where*

$$\text{D}(m) = a_1 a_2 a_3 (c_1 c_2 c_3)^2 \det(m)^4. \quad \square$$

Note that this result was obtained by Edge [4], in a more intricate way.

We denote by \mathbf{S} the set of $m \in \mathbf{Sym}_3(K)$ such that

$$a_1 a_2 a_3 \neq 0, \quad c_1 c_2 c_3 \neq 0.$$

Now, Prop. 2.2.1 implies that the curve X_m is nonsingular if and only if m belongs to the set $\mathbf{S}^\times = \mathbf{S} \cap \text{GL}_3(K)$.

Lemma 2.2.2. *The map $m \mapsto Q_m$ from \mathbf{S}^\times to the set \mathbf{Q} of Ciani forms is a bijection.* \square

The automorphisms of X_m induce a simple description of its Jacobian. In order to make it explicit, we need to introduce a certain product of elliptic curves.

2.3. Product of elliptic curves. We introduce the following notations : let

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i), \quad (b_i \in K, c_i \in K^\times) \quad (i = 1, 2, 3),$$

three elliptic curves with $(0, 0)$ as a rational 2-torsion point. The discriminant of E_i is $\Delta_i = 2^{12} c_i^2 \delta_i$, where $\delta_i = b_i^2 + c_i \in K^\times$. We assume that there exists a square root $\rho \in K^\times$ of $\delta(A) = \delta_1 \delta_2 \delta_3$, that is, $\Delta_1 \Delta_2 \Delta_3$ is a square in K . We denote by \mathbf{A} the set of products $E_1 \times E_2 \times E_3$ of such curves and we define

$$\tilde{\mathbf{A}} = \left\{ \tilde{A} = (A, \rho) \in \mathbf{A} \times K^\times \mid \rho^2 = \delta(A) \right\}.$$

If $\tilde{A} \in \tilde{\mathbf{A}}$, we put $a_i = \rho / \delta_i$ and

$$\mathbf{Mat}(\tilde{A}) = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \in \mathbf{S}.$$

Conversely, a matrix $m \in \mathbf{S}$ defines an abelian threefold $A(m) \in \mathbf{A}$, which is the product of the curves

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i) \quad (i = 1, 2, 3).$$

Then

$$\delta_i = b_i^2 + c_i = a_j a_k \in K^\times, \quad \Delta_1 \Delta_2 \Delta_3 = (2^{18} a_1 a_2 a_3 c_1 c_2 c_3)^2, \quad \delta(A) = (a_1 a_2 a_3)^2.$$

We define $\rho(m) = a_1 a_2 a_3$, and $\mathbf{Ab}(m) = (A(m), \rho(m)) \in \tilde{\mathbf{A}}$.

Lemma 2.3.1. *The maps*

$$\mathbf{Mat}: \tilde{\mathbf{A}} \longrightarrow \mathbf{S}, \quad \mathbf{Ab}: \mathbf{S} \longrightarrow \tilde{\mathbf{A}},$$

are mutually inverse bijections. \square

The two lemmas 2.2.2 and 2.3.1 provide a natural map from the set \mathbf{Q} of Ciani quartic forms to $\tilde{\mathbf{A}}$. This map has actually a geometric meaning, and in order to explain it, we introduce the following notation. If $m \in \mathrm{GL}_3(K)$, we denote by $\mathrm{Cof}(m)$ the *cofactor matrix* of m , satisfying

$$m \cdot {}^t \mathrm{Cof} m = \det m \cdot \mathbf{I}, \quad \det \mathrm{Cof} m = (\det m)^2, \quad \mathrm{Cof} \mathrm{Cof} m = (\det m) \cdot m.$$

Let Q_m be a Ciani form associated to $m \in \mathbf{S}^\times$ and X_m be the corresponding Ciani quartic

$$X_m : Q_m(x, y, z) = F_m(x^2, y^2, z^2) = 0.$$

By quotient, we get three genus one curves

$$\begin{aligned} C_1 &:= X_m / \langle 1, \sigma_1 \rangle : F(yz, x^2, y^2) = 0, \\ C_2 &:= X_m / \langle 1, \sigma_2 \rangle : F(zx, y^2, z^2) = 0, \\ C_3 &:= X_m / \langle 1, \sigma_3 \rangle : F(xy, z^2, x^2) = 0, \end{aligned}$$

where σ_i ($i = 1, 2, 3$) are the involutions of X_m . Another change of variables maps the genus 1 quartics C_i to the elliptic curves

$$F_i : y^2 = x(x^2 - 4d_i x - 4a_i \det(m)), \quad (i = 1, 2, 3).$$

In this way, we get a map

$$\varphi : X_m \longrightarrow B_m = F_1 \times F_2 \times F_3.$$

Let us now look more closely at B_m . The identity

$$\mathrm{Cof} \mathrm{Cof} m = (\det m) \cdot m$$

implies that the cofactor of c_i is $a_i \det m$. Hence,

$$\mathbf{Ab}(\mathrm{Cof} m) = (B_m, c_1 c_2 c_3).$$

Since the Jacobian is the Albanese variety of X_m , we get a factorization

$$\begin{array}{ccc} X_m & & \\ \iota \downarrow & \searrow \varphi & \\ \mathrm{Jac} X_m & \xrightarrow{\Phi} & B_m \end{array}$$

where ι is a canonical embedding. Since the images of the regular differential forms on F_i make a basis of those on X_m , we obtain:

Proposition 2.3.2. *The map*

$$\Phi : \mathrm{Jac} X_m \longrightarrow A(\mathrm{Cof} m)$$

is a $(2, 2, 2)$ -isogeny defined over K . \square

The correspondences

$$\begin{array}{ccccc} m & \longrightarrow & \mathrm{Cof} \, m & & \\ \downarrow & & \downarrow & & \\ Q_m & \longrightarrow & A(\mathrm{Cof} \, m) & \xrightarrow{\mathrm{isg}} & \mathrm{Jac} \, X_m \end{array}$$

lead to a commutative diagram, where \mathbf{Q} is the space of Ciani quartics over K :

$$\begin{array}{ccc} \mathbf{S}^\times & \xrightarrow{\mathrm{Cof}} & \mathbf{S}^\times \\ \downarrow = & & \downarrow \mathbf{Ab} \\ \mathbf{Q} & \xrightarrow{\text{"Jac"}} & \tilde{\mathbf{A}} \end{array}$$

In the next section we describe the kernel of the isogeny Φ .

2.4. The theory of Howe, Leprevost and Poonen revisited. The previous isogeny can be made more precise as we recall from [8]. There are some differences between their notation and ours, see the remark at the end of Sec.2.5 for a comparison. Let us introduce couples (A, W) , where:

- (i) $A \in \mathbf{A}$ as defined in § 2.3.

The Weil pairings on the factors combine to give a non degenerate alternating pairing e_2 on the finite group scheme $A[2]$ over K .

- (ii) W is a totally isotropic indecomposable subspace of $A[2]$ defined over K .

Choose a basis $(P_i, Q_i) \in E_i[2]$, that is, a level 2 structure on E_i . This defines a level 2 structure on A . In [8, Lem.13] it is proved that after a labeling of the 2-torsion points we can write

$$(2) \quad W = \left\{ \begin{array}{cccc} (O, O, O), & (O, Q_2, Q_3), & (Q_1, O, Q_3), & (Q_1, Q_2, O), \\ (P_1, P_2, P_3), & (P_1, R_2, R_3), & (R_1, P_2, R_3), & (R_1, R_2, P_3) \end{array} \right\}$$

with

$$Q_i = (0, 0), \quad P_i = (0, 2b_i + \rho_i), \quad R_i = (0, 2b_i - \rho_i), \quad \rho_1 \rho_2 \rho_3 = \rho_W,$$

and the four possible choices of ρ_1, ρ_2, ρ_3 leading to the same value of $\rho_1 \rho_2 \rho_3$ give the same subgroup W . Conversely, if $(A, \rho) \in \tilde{\mathbf{A}}$ is given, if we choose ρ_1, ρ_2, ρ_3 in such a way that $\rho_1 \rho_2 \rho_3 = \rho$, and if we define P_i and R_i as above, then we can define a subgroup W_ρ by (2).

Lemma 2.4.1. *The map $(A, \rho) \mapsto (A, W_\rho)$ from $\tilde{\mathbf{A}}$ to the set of couples (A, W) as defined above is a bijection. \square*

We take on $A = A(m)$ the principal polarization λ which is the product of the canonical polarizations on each factor. Then we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{2\lambda} & A^\vee \\ \pi \downarrow d^\circ 8 & & d^\circ 8 \uparrow \hat{\pi} \\ A' & \xrightarrow{\lambda'} & (A')^\vee \end{array}$$

with a unique principal polarization λ' on $A' = A'(m) = A(m)/W_{\rho(m)}$. From [8, Prop.15] we get :

Theorem 2.4.2. *The composition of the isogeny Φ and of the projection π leads to an isomorphism of p.p.a.v.:*

$$\mathrm{Jac} \, X_m \longrightarrow A'(\mathrm{Cof} \, m). \quad \square$$

As a corollary, for any m , the p.p.a.v. (A', λ') is indecomposable.

The reverse direction is more interesting and will give an algebraic answer to Serre's conjecture.

2.5. Relation with Serre's conjecture. We need the following elementary lemma from linear algebra.

Lemma 2.5.1. *The map $m \mapsto \text{Cof } m$ induces an exact sequence*

$$1 \longrightarrow \{\pm 1\} \longrightarrow \text{GL}_3(K) \xrightarrow{\text{Cof}} G^{\times 2}(K) \longrightarrow 1$$

with

$$G^{\times 2}(K) = \{m \in \text{GL}_3(K) \mid \det m \in K^{\times 2}\}. \quad \square$$

Let $(A, \rho) = \tilde{A} \in \tilde{\mathcal{A}}$ and $m = \mathbf{Mat}(\tilde{A})$. Denote

$$\mathbf{T}(\tilde{A}) := \det(m) = 2b_1b_2b_3 - \rho\left(\frac{b_1^2}{\delta_1} + \frac{b_2^2}{\delta_2} + \frac{b_3^2}{\delta_3} - 1\right).$$

Theorem 2.5.2. *The following results hold.*

- (i) *if $\mathbf{T}(\tilde{A}) = 0$, that is, $m \in \mathcal{S} \setminus \mathcal{S}^\times$, there is a hyperelliptic curve X of genus 3 such that $A'(m)$ is isomorphic to the Jacobian of X .*
- (ii) *if $\mathbf{T}(\tilde{A}) \neq 0$, that is, $m \in \mathcal{S}^\times$, then there exists a non hyperelliptic curve of genus 3 defined over K whose Jacobian is isomorphic to $A'(m)$ if and only if $\mathbf{T}(\tilde{A})$ is a square in K .*

Proof. The first part is [8, Prop.14] where the hyperelliptic curve is constructed explicitly. For the second part, if $\det(m)$ is a square, then using Lem. 2.5.1, we see that there exists a matrix $m' \in \mathcal{S}^\times$ such that $m = \text{Cof}(m')$ and we apply Th. 2.4.2. If $d = \det(m)$ is not a square, let $m_d = dm$ and $\tilde{A}_d = \mathbf{Ab}(m_d)$. $A(m_d)$ is defined by

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i d^2) \quad (i = 1, 2, 3),$$

Thus $A(m_d)$ is a quadratic twist of $A(m)$. Now, $\det(m_d)$ is a square, so there exists m' such that $\text{Jac}(X_{m'})$ is isomorphic to $A'(m_d)$. Since $A'(m_d)$ is a quadratic twist of $A'(m)$ and is the Jacobian of a non hyperelliptic curve, Th. 1.2.1 shows that $A'(m)$ cannot be a Jacobian. \square

Corollary 2.5.3. *With the same notation as above:*

- (i) *if $\mathbf{T}(\tilde{A}) \in K^{\times 2}$, there is an isogeny defined over K*

$$\text{Jac } X_{m'} \longrightarrow A(m), \quad \text{Cof } m' = m,$$

- (ii) *If $\mathbf{T}(\tilde{A}) \notin K^{\times 2}$, there is an isogeny defined over K*

$$\text{Jac } X_{m'} \longrightarrow A(m_d) \quad \text{Cof } m' = dm. \quad \square$$

We hope to give in a near future a geometric interpretation of the connection of Serre's problem with the determinant of certain quadratic forms in the general case.

Remark. In [8], Howe, Leprevost and Poonen write the elliptic curves

$$y^2 = x(x^2 + A_i x + B_i) \quad \text{avec } A_i, B_i \in K \quad (i = 1, 2, 3).$$

So

$$\begin{aligned} A_i &= -4b_i, \quad B_i = -4c_i, \\ \Delta_i &= A_i^2 - 4B_i = 16(b_i^2 + c_i) = 16\delta_i, \\ d_i &= -(A_i + 2x(P_i)) = 4b_i - 4(b_i + \rho_i) = -4\rho_i, \quad d_i^2 = \Delta_i. \end{aligned}$$

And the factor

$$T_0(\tilde{A}) = d_1 d_2 d_3 \left(\frac{A_1^2}{\Delta_1} + \frac{A_2^2}{\Delta_2} + \frac{A_3^2}{\Delta_3} - 1 \right) - 2A_1 A_2 A_3,$$

which is

$$T_0(\tilde{A}) = 64[-\rho(\frac{b_1^2}{\delta_1} + \frac{b_2^2}{\delta_2} + \frac{b_3^2}{\delta_3} - 1) + 2b_1b_2b_3].$$

We then have $T_0(\tilde{A}) = 64 \mathbf{T}(\tilde{A})$.

3. COMPLEX ABELIAN VARIETIES

We recall in this section some well known propositions on abelian varieties over \mathbb{C} and fix the notation.

3.1. The symplectic group. If V is a module of rank $2g$ over a commutative ring R and if E is a non-degenerate alternating bilinear form on V , a basis $(a_i)_{1 \leq i \leq 2g}$ of V is said *symplectic* if the matrix $(E(a_i, a_j)) = J$, where

$$J = \begin{bmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{bmatrix}.$$

The group of matrices

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{SL}_{2g}(R)$$

such that $M.J.^tM = J$ is the *symplectic group* $\mathrm{Sp}_{2g}(R)$. It acts simply transitively on the set of symplectic bases of V .

Lemma 3.1.1 ([2, Lem.8.2.1]). *If $M \in \mathrm{Sp}_{2g}(R)$ the following conditions are equivalent.*

- (i) $M \in \mathrm{Sp}_{2g}(R)$.
- (ii) ${}^tA.C$ and ${}^tB.D$ are symmetric, and ${}^tA.D - {}^tC.B = \mathbf{1}_g$.
- (iii) $A{}^tB$ and $C{}^tD$ are symmetric and $A.{}^tD - B.{}^tC = \mathbf{1}_g$. □

The group $\mathrm{Sp}_{2g}(R)$ is the group of R -rational points of a Chevalley group scheme Sp_{2g} , which contains certain remarkable subgroups defined as follows. The reductive subgroup \mathbf{M} of Sp_{2g} is the subgroup which respects the canonical decomposition $\mathbb{Z}^{2g} = \mathbb{Z}^g \oplus \mathbb{Z}^g$. Elements of $\mathbf{M}(\mathbb{Z})$ are

$$M = \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}, \quad A \in \mathrm{GL}_g(\mathbb{Z}).$$

The unipotent subgroup \mathbf{U} is the stability group leaving pointwise fixed the canonical totally isotropic subspace V_0 , which is the first direct summand in the standard decomposition. Elements of $\mathbf{U}(\mathbb{Z})$ are

$$U = \begin{bmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{bmatrix}, \quad {}^tB = B, \quad B \in \mathbf{M}_g(\mathbb{Z}).$$

The unipotent subgroup \mathbf{V} opposite to \mathbf{U} is the stability group of the second direct summand in the standard decomposition. One has $\mathbf{V} = {}^t\mathbf{U} = J.\mathbf{U}.J^{-1}$. Elements of $\mathbf{V}(\mathbb{Z})$ are

$$V = \begin{bmatrix} \mathbf{1}_g & 0 \\ C & \mathbf{1}_g \end{bmatrix}, \quad {}^tC = C, \quad C \in \mathbf{M}_g(\mathbb{Z}).$$

The subgroup $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ is the parabolic subgroup of Sp_{2g} normalizing V_0 , and \mathbf{P} is actually a maximal parabolic subgroup. Elements of $\mathbf{P}(\mathbb{Z})$ are

$$P = \begin{bmatrix} A & B \\ 0 & {}^tA^{-1} \end{bmatrix}, \quad A.{}^tB = B.{}^tA, \quad A \in \mathrm{GL}_g(\mathbb{Z}), \quad B \in \mathbf{M}_g(\mathbb{Z}).$$

3.2. Abelian varieties. Let $\Omega = [w_1 \dots w_{2g}] \in \mathbf{M}_{g,2g}(\mathbb{C})$, where w_1, \dots, w_{2g} are columns vectors giving a basis of \mathbb{C}^g on \mathbb{R} . It generates a lattice

$$\Lambda = \Omega \mathbb{Z}^{2g} \subset \mathbb{C}^g.$$

Let \mathcal{R}_g be the set of matrices $\Omega \in \mathbf{M}_{g,2g}(\mathbb{C})$ satisfying the *Riemann conditions*

$$\Omega \cdot J \cdot \Omega = 0, \quad 2i(\overline{\Omega} \cdot J^{-1} \cdot \Omega)^{-1} > 0$$

(> 0 means positive definite). We call such a matrix Ω a *period matrix*. If $\Omega \in \mathcal{R}_g$, the torus $A_\Omega = \mathbb{C}^g / \Lambda$ is an abelian variety of dimension g with a principal polarization λ represented by the hermitian form $H = 2i(\overline{\Omega} \cdot J^{-1} \cdot \Omega)^{-1}$ (see [2, Lem.4.2.3]).

The group $\mathrm{GL}_g(\mathbb{C})$ acts on the left on \mathcal{R}_g . If we write

$$\Omega = [(w_1 \dots w_g) (w_{g+1} \dots w_{2g})] = [\Omega_1 \ \Omega_2], \quad \text{where } \Omega_i \in \mathbf{M}_g(\mathbb{C}),$$

we get $W \cdot [\Omega_1 \ \Omega_2] = [W \cdot \Omega_1 \ W \cdot \Omega_2]$ for any $W \in \mathrm{GL}_g(\mathbb{C})$. This action induces an isomorphism of p.p.a.v. In particular if we choose $W = \Omega_2^{-1}$, we see that A_Ω is isomorphic to the p.p.a.v.

$$A_\tau = A_{\Omega(\tau)}, \quad \Omega(\tau) = [\tau \ \mathbf{1}_g], \quad \tau = \tau(\Omega) = \Omega_2^{-1} \Omega_1,$$

and $\Omega \in \mathcal{R}_g$ if and only if $\tau(\Omega)$ belongs to the Siegel upper half plane

$$\mathbb{H}_g = \{ \tau \in \mathbf{M}_g(\mathbb{C}) \mid {}^t \tau = \tau, \operatorname{Im} \tau > 0 \}.$$

We call a matrix $\tau \in \mathbb{H}_g$ a *Riemann matrix*. The *Siegel modular group* $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the right on \mathcal{R}_g : if $\Omega \in \mathcal{R}_g$ and if $M \in \Gamma_g$,

$$\Omega \cdot M = [\Omega_1 \ \Omega_2] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\Omega_1 A + \Omega_2 C \ \Omega_1 B + \Omega_2 D].$$

This action corresponds to a change of symplectic basis. The group Γ_g also acts on the left on the Siegel upper half plane : if $\tau \in \mathbb{H}_g$, we denote

$$M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.$$

Both actions are linked by

$$(3) \quad M \cdot \tau(\Omega) = \tau(\Omega \cdot {}^t M).$$

3.3. Isotropy and quotients. For any maximal isotropic subgroup $V \subset \mathbb{F}_2^{2g}$, we have the *transporter*

$$\mathrm{Trans}(V) = \{ M \in \mathrm{Sp}_{2g}(\mathbb{F}_2) \mid MV_0 = V \},$$

V_0 being the canonical maximal isotropic subgroup generated by the vectors e_1, \dots, e_g of the canonical basis. Since $\mathrm{Sp}_{2g}(\mathbb{F}_2)$ permutes transitively the maximal isotropic subgroups of \mathbb{F}_2^{2g} , the transporter is a left coset: $\mathrm{Trans}(V) = M_0 \mathbf{P}(\mathbb{F}_2)$, for any $M_0 \in \mathrm{Trans}(V)$. Hence, the set of maximal isotropic subgroups is the quotient set $\mathrm{Sp}_{2g}(\mathbb{F}_2) / \mathbf{P}(\mathbb{F}_2)$, a set with 135 elements if $g = 3$.

Let now $\Omega \in \mathcal{R}_g$, $\Lambda = \Omega \mathbb{Z}^{2g}$ and $(A, \lambda) = (A_\Omega, H)$ be the corresponding p.p.a.v. of dimension g . The linear map $\alpha : \mathbb{Z}^{2g} \rightarrow \frac{1}{2}\Lambda$ such that

$$\alpha(x) = \frac{1}{2} \Omega \cdot x$$

defines a level 2 symplectic structure on $A[2]$, that is, an isomorphism

$$\bar{\alpha} : \mathbb{F}_2^{2g} \xrightarrow{\sim} A[2]$$

and if $V \subset \mathbb{F}_2^{2g}$ is a maximal isotropic subgroup, the same property holds for $W = \bar{\alpha}(V) \subset A[2]$. If $\pi : \mathbb{C}^3 \rightarrow \mathbb{C}^3/\Lambda$ is the canonical projection, the lattice $\Lambda_W = \pi^{-1}(W)$ is associated to A/W as the following diagram shows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda_W/\Lambda & \longrightarrow & \mathbb{C}^3/\Lambda & \longrightarrow & \mathbb{C}^3/\Lambda_W \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & W & \longrightarrow & A & \longrightarrow & A/W \longrightarrow 0. \end{array}$$

We define

$$\text{Trans}(W) = \{M \in \Gamma_g \mid M(\bmod 2) \in \text{Trans}(V)\}.$$

We introduce now the congruence subgroup

$$\Gamma_{0,g}(2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g \mid C \equiv 0(\bmod 2) \right\}.$$

This is the transposed subgroup of the group $\Gamma_g^0(2)$, see § A.1. From Prop. A.1.1 we deduce that $\Gamma_{0,g}(2) = \mathbf{P}(\mathbb{Z}).\Gamma_g(2)$, hence

$$\text{Trans}(W) = M\Gamma_{0,g}(2)$$

for any $M \in \text{Trans}(W)$.

Proposition 3.3.1. *With the previous notation, if $\tau = \tau(\Omega)$ then $\frac{1}{2}{}^t M.\tau$ is a Riemann matrix of the p.p.a.v. A/W for all $M \in \text{Trans}(W)$.*

Proof. If $M \in \text{Trans}(W)$, $W \bmod \Lambda$ is generated by the vectors

$$\frac{1}{2}\Omega.Me_1, \quad \dots \quad \frac{1}{2}\Omega.Me_g,$$

and the matrix

$$\Omega' = \Omega.M.H, \quad H = \begin{bmatrix} \frac{1}{2}\mathbf{1}_g & 0 \\ 0 & \mathbf{1}_g \end{bmatrix},$$

generates Λ_W . Using (3), we get

$$\tau(\Omega') = {}^t(M.H).\tau = \frac{1}{2}{}^t M.\tau.$$

By [14, Prop. 16.8], the polarization 2λ of A reduces to a principal polarization λ' on $A' = A/W$. This last corresponds canonically to Ω' since

$$2i(\overline{\Omega'}.J^{-1}.{}^t\Omega')^{-1} = 2i(\overline{\Omega}MHJ^{-1}{}^tH{}^tM{}^t\Omega)^{-1} = 2 \cdot 2i(\overline{\Omega}J^{-1}{}^t\Omega)^{-1}.$$

□

3.4. Theta functions. We recall the definition of theta functions with (entire) characteristics $[\varepsilon] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$ where $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}^g$, following [2]. The (classical) theta function is

$$\vartheta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}^g} q^{(n+\varepsilon/2)\tau(n+\varepsilon/2)+2(n+\varepsilon/2)(z+\varepsilon_2/2)} \quad (\tau \in \mathbb{H}_g, z \in \mathbb{C}^g).$$

The *Thetanullwerte* are the values at $z = 0$ of these functions, and we denote

$$\vartheta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (\tau) = \vartheta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (0, \tau).$$

We now state two formulas.

Proposition 3.4.1 (duplication formula, see [16, Cor.IIA2.1] and [10, IV.th.2]). *Let $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$ and $\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$ be two characteristics and $\tau \in \mathbb{H}_g$. Then*

$$(4) \quad \vartheta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (\tau/2) \vartheta \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} (\tau/2) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^g} (-1)^{\mu\delta} \cdot \vartheta \begin{bmatrix} \varepsilon_1 - \mu \\ \varepsilon_2 - \delta \end{bmatrix} (\tau) \cdot \vartheta \begin{bmatrix} \mu \\ \varepsilon_2 - \delta \end{bmatrix} (\tau).$$

The second is called *transformation formula*. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$. We let M acts on the characteristics in the following way

$$[M.\varepsilon] = M. \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} D\varepsilon_1 - C\varepsilon_2 + (C^t D)_0 \\ -B\varepsilon_1 + A\varepsilon_2 + (A^t B)_0 \end{bmatrix}$$

where P_0 denotes the diagonal of the matrix P .

Proposition 3.4.2 ([10, V.§.2]).

$$\vartheta[M.\varepsilon](M.\tau) = \kappa(M) \cdot \omega^{\phi_{[\varepsilon_1, \varepsilon_2]}(M)} \cdot j(M, \tau)^{1/2} \cdot \vartheta[\varepsilon](\tau)$$

where $\kappa(M)^2$ is a root of 1 depending only on M , $\omega = e^{i\pi/4}$,

$$j(M, \tau) = \det(C\tau + D)$$

and

$$\phi_{[\varepsilon_1, \varepsilon_2]}(M) = \varepsilon_1 {}^t D B \varepsilon_1 - 2\varepsilon_1 {}^t B C \varepsilon_2 + \varepsilon_2 {}^t C A \varepsilon_2 - 2(D\varepsilon_1 - C\varepsilon_2) \cdot (A^t B)_0. \quad \square$$

We will need a slightly modified version of the previous result.

Corollary 3.4.3. For any characteristic $\begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \end{bmatrix}$ and for any $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$ we have

$$(5) \quad \vartheta \begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \end{bmatrix} (M.\tau) = c(M, \tau) \cdot \omega^{-\phi_{[\varepsilon'_1, \varepsilon'_2]}(M^{-1})} \cdot \vartheta \begin{bmatrix} {}^t A(\varepsilon'_1 - (C^t D)_0) + {}^t C(\varepsilon'_2 - (A^t B)_0) \\ {}^t B(\varepsilon'_1 - (C^t D)_0) + {}^t D(\varepsilon'_2 - (A^t B)_0) \end{bmatrix} (\tau)$$

where

$$c(M, \tau) = \kappa(M^{-1})^{-1} \cdot j(M, \tau).$$

Proof. To inverse the action on the characteristics, we let $\tau' = M^{-1}.\tau$ in the transformation formula. Note that

$$M^{-1} = \begin{pmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{pmatrix}$$

and that $[M.\varepsilon] = [{}^t M^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}] + \begin{bmatrix} (C^t D)_0 \\ (A^t B)_0 \end{bmatrix}$. Thus we get the action on the characteristics. For the factor $j(M, \tau)$ note that

$$\begin{aligned} j(M, \tau) &= \det(C\tau + D) = \det(CM^{-1}.\tau' + D) \\ &= \det(C({}^t D\tau' - {}^t B)(-{}^t C\tau' + {}^t A)^{-1} + D) \\ &= \det(C({}^t D\tau' - {}^t B) + D(-{}^t C\tau' + {}^t A)) \det(-{}^t C\tau' + {}^t A)^{-1} \\ &= \det(-{}^t C\tau' + {}^t A)^{-1} = j(M^{-1}, \tau')^{-1} \end{aligned}$$

using Lem. 3.1.1. \square

Corollary 3.4.4. Let $\Omega = [\Omega_1 \ \Omega_2]$ be a period matrix and $\tau = \tau(\Omega) = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g$. Let $\Omega' = \Omega^t M = [\Omega'_1 \ \Omega'_2]$. Then

$$j(M, \tau(\Omega)) = \det(\Omega_2)^{-1} \cdot \det(\Omega'_2).$$

Proof. We compute

$$\begin{aligned} \det(C\tau + D) &= \det(\tau {}^t C + {}^t D) \\ &= \det(\Omega_2)^{-1} \det(\Omega_1 {}^t C + \Omega_2 {}^t D) \\ &= \det(\Omega_2)^{-1} \cdot \det(\Omega'_2), \end{aligned}$$

the last expression coming from (3). \square

3.5. The modular function χ_k . Recall that a characteristic $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$ is *even* (resp. *odd*) if $\varepsilon_1, \varepsilon_2 \equiv 0 \pmod{2}$ (resp. $\varepsilon_1, \varepsilon_2 \equiv 1 \pmod{2}$). Let S_g (resp. U_g) be the set of even (resp. odd) characteristics $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$ with coefficients in $\{0, 1\}$. It is well known that

$$\#S_g = 2^{g-1}(2^g + 1), \quad \#U_g = 2^{g-1}(2^g - 1).$$

Let $\Omega = [\Omega_1 \ \Omega_2] \in \mathcal{R}_g$ and $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g$ be a Riemann matrix. For $g \geq 2$, we denote $k = \#S_g/2$ and we are interested in the following expressions :

$$\chi_k(\tau) = \prod_{\varepsilon \in S_g} \vartheta[\varepsilon](\tau).$$

Recall that a function f is a *modular form* of weight w for the congruence subgroup $\Gamma \in \Gamma_g$ if for all $\tau \in \mathbb{H}_g$ and $M \in \Gamma$ one has

$$f(M.\tau) = j(M, \tau)^w f(\tau).$$

Using Cor.3.4.4, we get

Corollary 3.5.1. *Let f be a modular form of weight k for Γ on \mathbb{H}_g . For $\Omega = [\Omega_1 \ \Omega_2] \in \mathcal{R}_g$, we define $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g$ a Riemann matrix and*

$$f(\Omega) := \det(\Omega_2)^{-k} \cdot f(\tau).$$

Then for all $M \in \Gamma$

$$f(\Omega.M) = f(\Omega). \quad \square$$

In his beautiful paper [9], Igusa proves the following result [*loc. cit.*, Lem. 10 & 11]. Denote by Σ_{140} the thirty-fifth elementary symmetric function of the eighth power of the even Thetanullwerte.

Theorem 3.5.2. *For $g \geq 3$, the product $\chi_k(\tau)$ is a modular form of weight k for the group Γ_g . Moreover, If $g = 3$ and $\tau \in \mathbb{H}_3$, then:*

- (i) A_τ is decomposable if $\chi_{18}(\tau) = \Sigma_{140}(\tau) = 0$.
- (ii) A_τ is a hyperelliptic Jacobian if $\chi_{18}(\tau) = 0$ and $\Sigma_{140}(\tau) \neq 0$.
- (iii) A_τ is a non hyperelliptic Jacobian if $\chi_{18}(\tau) \neq 0$. \square

This theorem gives an answer to the two questions raised in Sec.1.1 over \mathbb{C} .

In the sequel, we will need the following result to prove the independence of our results from the choices we will make. The proof is the case $g = 3$ of Th. A.1.2.

Proposition 3.5.3. *The product $\tau \mapsto \chi_{18}(\frac{1}{2}\tau)$ is a modular form on \mathbb{H}_3 of weight 18 for $\Gamma_3^0(2)$.*

4. COMPARISON OF ANALYTIC AND ALGEBRAIC DISCRIMINANTS

In this part, we make the link between the algebraic result Th.2.5.2 and Serre's conjecture on the modular function χ_{18} . To do so, we first compute a quantity related easily to $T(\tilde{A})$ in terms of the Thetanullwerte on the elliptic curves. Then, after a good choice of a symplectic matrix N (related to the subgroup W we use for the quotient), we compute $\chi_{18}(({}^tN.\tau)/2)$ in terms of the same Thetanullwerte. Thus, we express χ_{18} on the quotient A/W . Finally we compare the expressions to get Serre's conjecture.

4.1. Expression of the algebraic discriminant. We come back to the hypotheses of § 2.3, and specialize to the case $K \subset \mathbb{C}$. Let $A = E_1 \times E_2 \times E_3 \in \mathbf{A}$, where

$$E_i : y^2 = x(x^2 - 4b_i x - 4c_i), \quad (b_i \in K, c_i \in K^\times) \quad (i = 1, 2, 3).$$

We choose a root ρ of $\delta(A)$, and put $m = \mathbf{Mat}(\tilde{A})$ with $\tilde{A} = (A, \rho)$. Let

$$X(m) := (a_1 a_2 a_3)^4 (c_1 c_2 c_3)^2 \det m.$$

Since $\mathbf{T}(\tilde{A}) = \det m$, $\mathbf{T}(\tilde{A})$ is a square in K if and only if $X(m)$ is a square in K . The function X appears naturally in our problem since it is related to the function $D(m)$ (Prop. 2.2.1) by

$$(6) \quad X(\text{Cof } m) = D(m)^2,$$

and this reflects Serre's conjecture according to Th.2.4.2. In order to determine the expression of $X(m)$ in terms of the Thetanullwerte, we use the following uniformization. The curves E_i can be written as

$$E(\omega_{1i}, \omega_{2i}) : y^2 = x(x + \frac{\pi^2}{\omega_{2i}^2} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau_i)^4)(X + \frac{\pi^2}{\omega_{2i}^2} \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau_i)^4),$$

with $[\omega_{1i} \ \omega_{2i}] \in \mathcal{R}_1$ and $\tau_i = \frac{\omega_{1i}}{\omega_{2i}}$. We identify \mathcal{R}_1^3 with the set of matrices

$$\Omega = [\Omega_1 \ \Omega_2] = \left[\begin{pmatrix} \omega_{11} & 0 & 0 \\ 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{13} \end{pmatrix} \begin{pmatrix} \omega_{21} & 0 & 0 \\ 0 & \omega_{22} & 0 \\ 0 & 0 & \omega_{23} \end{pmatrix} \right]$$

such that

$$\tau = \tau(\Omega) = \Omega_2^{-1} \Omega_1 = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \in \mathbb{H}_3.$$

We define

$$A(\Omega) := E(\omega_{11}, \omega_{21}) \times E(\omega_{12}, \omega_{22}) \times E(\omega_{13}, \omega_{23}),$$

$$\rho(\Omega) := \frac{\pi^6}{64(\det \Omega_2)^2} \vartheta^4 \begin{bmatrix} 111 \\ 000 \end{bmatrix} (\tau).$$

This defines an element $\tilde{A}(\Omega) := (A(\Omega), \rho(\Omega)) \in \tilde{\mathbf{A}}$, and a matrix $m(\Omega) := \mathbf{Mat}(\tilde{A}(\Omega))$. For $1 \leq i \leq 3$, denote

$$\vartheta_{0i} = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau_i), \quad \vartheta_{1i} = \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau_i), \quad \vartheta_{2i} = \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau_i).$$

The coefficients of $A(\Omega)$ and $m(\Omega)$ are

$$a_i = -\frac{\pi^2}{4} \frac{\omega_{2i}^2}{(\omega_{2j}\omega_{2k})^2} \frac{\vartheta_{1j}^4 \vartheta_{1k}^4}{\vartheta_{1i}^4},$$

$$b_i = -\frac{\pi^2}{4\omega_{2i}^2} (\vartheta_{0i}^4 + \vartheta_{2i}^4),$$

$$c_i = -\frac{\pi^4}{4\omega_{2i}^4} \vartheta_{0i}^4 \vartheta_{2i}^4,$$

where (i, j, k) is a cyclic permutation. The determinant of $m(\Omega)$ is expressed as follows. Let

$$a = \vartheta_{01}^2 \vartheta_{02}^2 \vartheta_{23}^2, \quad b = \vartheta_{01}^2 \vartheta_{22}^2 \vartheta_{03}^2, \quad c = \vartheta_{21}^2 \vartheta_{02}^2 \vartheta_{03}^2, \quad d = \vartheta_{21}^2 \vartheta_{22}^2 \vartheta_{23}^2,$$

$$R_1 = (a + b + c + d)(a + b - c - d)(a - b - c + d)(a - b + c - d),$$

Then

$$\det m(\Omega) = \frac{\pi^6}{2^4 \cdot \prod_{i=1}^3 (\omega_{2i}^2 \cdot (\vartheta_{0i}^4 - \vartheta_{2i}^4))} \cdot R_1.$$

Thus we get

$$\begin{aligned}
\mathbf{X}(m(\Omega)) &= \left(\frac{\pi^{12}}{2^6 \cdot \prod_{i=1}^3 \omega_{2i}^4} \cdot \prod_{i=1}^3 \vartheta_{0i}^4 \vartheta_{2i}^4 \right)^2 \cdot \left(\frac{\pi^6}{2^6 \cdot \prod_{i=1}^3 \omega_{2i}^2} \cdot \prod_{i=1}^3 (\vartheta_{0i}^4 - \vartheta_{2i}^4) \right)^4 \\
&\quad \cdot \left(\frac{\pi^6}{2^4 \cdot \prod_{i=1}^3 (\omega_{2i}^2 \cdot (\vartheta_{0i}^4 - \vartheta_{2i}^4))} \cdot R_1 \right) \\
&= \frac{\pi^{54}}{2^{40}} \cdot \det(\Omega_2)^{-18} \cdot \left(\prod_{i=1}^3 \vartheta_{0i}^8 \vartheta_{2i}^8 (\vartheta_{0i}^4 - \vartheta_{2i}^4)^3 \right) \cdot R_1.
\end{aligned}$$

4.2. The subgroup W . With the notation of Sec. 2.4, we can always assume the following correspondences

$$P_i \leftrightarrow \frac{\omega_{1i}}{2}, \quad Q_i \leftrightarrow \frac{\omega_{2i}}{2}, \quad R_i = P_i + Q_i \leftrightarrow \frac{\omega_{1i} + \omega_{2i}}{2}$$

for the points of E_i . The characteristics associated to the points of W (see § 2.4) are

$$\begin{bmatrix} 000 \\ 000 \end{bmatrix}, \begin{bmatrix} 000 \\ 011 \end{bmatrix}, \begin{bmatrix} 000 \\ 101 \end{bmatrix}, \begin{bmatrix} 000 \\ 110 \end{bmatrix}, \begin{bmatrix} 111 \\ 000 \end{bmatrix}, \begin{bmatrix} 111 \\ 011 \end{bmatrix}, \begin{bmatrix} 111 \\ 101 \end{bmatrix}, \begin{bmatrix} 111 \\ 110 \end{bmatrix}.$$

It defines a maximal isotropic subgroup V of \mathbb{F}_2^6 . A basis of V over \mathbb{F}_2 is given by the three vectors

$$\alpha_1 = \begin{bmatrix} 000 \\ 011 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 000 \\ 110 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 111 \\ 000 \end{bmatrix}.$$

The matrix

$$N = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

belongs to Γ_3 and satisfies $N.e_i \equiv \alpha_i \pmod{2}$ if $1 \leq i \leq 3$, thus $N \in \text{Trans}(W)$. The set

$$\Gamma_g(1, 2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g \mid (A.^tB)_0 \equiv (C.^tD)_0 \equiv 0 \pmod{2} \right\},$$

is a subgroup of Γ_g , and κ^2 is a character of $\Gamma_g(1, 2)$ [10, p. 181].

Lemma 4.2.1. *The matrices N and tN are in $\Gamma_3(1, 2)$, and $\kappa(N)^2 = \kappa({}^tN)^2 = \pm 1$.*

Proof. We have $N = LQ$, where

$$L = \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}, \quad \text{with } A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

One checks easily that $L, {}^tL, Q, {}^tQ$ belong to $\Gamma_3(1, 2)$, hence, N and tN are in $\Gamma_3(1, 2)$ as well. If

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in \mathbf{P}(\mathbb{Z}),$$

then $\kappa(M)^2 = \det D$, see [9, Lem. 7, p. 181]. Now

$$Q^2 = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}, \quad \text{with } S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this we deduce that

$$\kappa(L)^2 = \det A = 1, \quad \kappa(Q)^4 = \kappa(Q^2)^2 = \det S = 1,$$

hence, $\kappa(N)^2 = \kappa({}^tN)^2 = \pm 1$. \square

Proposition 4.2.2. *Let $\Omega' = \Omega NH$. Then*

$$\tau(\Omega') = \frac{1}{2} {}^tN \cdot \tau$$

is a Riemann matrix for $A'(m)$. Moreover, the value $\chi_{18}(\Omega')$ is independent on the choice of $N \in \text{Trans}(W)$.

Proof. The first assertion comes from Prop. 3.3.1, the second from Prop. 3.5.3. \square

4.3. Expression of $\chi_{18}(\Omega')$ as a discriminant. Our main result in this section is the following

Theorem 4.3.1. *Let $\Omega \in \mathcal{R}_1^3$ and $A(\Omega)$ be the corresponding abelian threefold, let $m = m(\Omega) \in \mathcal{S}$ be the associated matrix, and $\Omega' \in \mathcal{R}_3$ be a Riemann matrix of $A(\Omega)/W$. Then*

$$\left(\frac{\pi}{2}\right)^{54} \cdot \chi_{18}(\Omega') = \mathbf{X}(m).$$

Proof. The strategy is the following. Let N be the matrix defined in § 4.2, and define $\tau' = {}^tN \cdot \tau = 2\tau(\Omega')$.

- (i) Pair the Thetanullwerte in $\tau'/2$ such that one can apply the duplication formula (4). We then obtain expressions in terms of Thetanullwerte in τ' . Such a pairing is not unique and one makes here a choice which allows an easy comparison of the final formulas.
- (ii) For each of the Thetanullwerte in τ' , apply the transformation formula (5) to obtain an expression in τ .
- (iii) Finally, since $\tau = \text{diag}(\tau_1, \tau_2, \tau_3)$, we get

$$\vartheta \begin{bmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \end{bmatrix} (\tau) = \prod_{i=1}^3 \vartheta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (\tau_i).$$

Let

$$c(N) = \kappa({}^tN^{-1})^{-2} \det(\Omega_2)^{-1} \det(\Omega'_2) = \pm \det(\Omega_2)^{-1} \det(\Omega'_2)$$

by Lem. 4.2.1.

Applying steps (i) to (iii) with the software MAGMA (see <http://iml.univ-mrs.fr/~ritzenh/programme/check>) we get the following 18 identities, where we write

$$\vartheta \begin{bmatrix} 000 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 000 \\ 001 \end{bmatrix} = \vartheta \begin{bmatrix} 000 \\ 000 \end{bmatrix} (\tau'/2) \vartheta \begin{bmatrix} 000 \\ 001 \end{bmatrix} (\tau'/2), \quad c = c(N).$$

We make the pairing in such a way that the expressions of $\vartheta \begin{bmatrix} 000 \\ \varepsilon_2 \end{bmatrix} \vartheta \begin{bmatrix} 000 \\ \delta \end{bmatrix}$ do not contain ϑ_{1i} terms. The first four are, with the preceding notation,

$$\begin{aligned} \vartheta \begin{bmatrix} 000 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 000 \\ 001 \end{bmatrix} &= c(a + b + c + d) \\ \vartheta \begin{bmatrix} 000 \\ 010 \end{bmatrix} \vartheta \begin{bmatrix} 000 \\ 011 \end{bmatrix} &= c(a + b - c - d) \\ \vartheta \begin{bmatrix} 000 \\ 100 \end{bmatrix} \vartheta \begin{bmatrix} 000 \\ 101 \end{bmatrix} &= -c(a - b - c + d) \\ \vartheta \begin{bmatrix} 000 \\ 110 \end{bmatrix} \vartheta \begin{bmatrix} 000 \\ 111 \end{bmatrix} &= -c(a - b + c - d) \end{aligned}$$

and the remaining 14 are

$$\vartheta \begin{bmatrix} 010 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 010 \\ 001 \end{bmatrix} = 2c(\vartheta_{01}\vartheta_{21}\vartheta_{02}\vartheta_{22}\vartheta_{03}^2 + \vartheta_{01}\vartheta_{21}\vartheta_{02}\vartheta_{22}\vartheta_{23}^2)$$

$$\begin{aligned}
\vartheta \begin{bmatrix} 100 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 100 \\ 001 \end{bmatrix} &= 2c(\vartheta_{01}^2 \vartheta_{02} \vartheta_{22} \vartheta_{03} \vartheta_{23} + \vartheta_{21}^2 \vartheta_{02} \vartheta_{22} \vartheta_{03} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 110 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 110 \\ 001 \end{bmatrix} &= 2c(\vartheta_{01}^2 \vartheta_{21} \vartheta_{02}^2 \vartheta_{03} \vartheta_{23} + \vartheta_{01} \vartheta_{21} \vartheta_{22}^2 \vartheta_{03} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 010 \\ 100 \end{bmatrix} \vartheta \begin{bmatrix} 010 \\ 101 \end{bmatrix} &= 2c(\vartheta_{01}^2 \vartheta_{21} \vartheta_{02} \vartheta_{22} \vartheta_{03}^2 - \vartheta_{01}^2 \vartheta_{21} \vartheta_{02} \vartheta_{22} \vartheta_{23}^2) \\
\vartheta \begin{bmatrix} 100 \\ 010 \end{bmatrix} \vartheta \begin{bmatrix} 100 \\ 011 \end{bmatrix} &= 2c(\vartheta_{01}^2 \vartheta_{02} \vartheta_{22} \vartheta_{03} \vartheta_{23} - \vartheta_{21}^2 \vartheta_{02} \vartheta_{22} \vartheta_{03} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 110 \\ 110 \end{bmatrix} \vartheta \begin{bmatrix} 110 \\ 111 \end{bmatrix} &= -2c(\vartheta_{01} \vartheta_{21} \vartheta_{02}^2 \vartheta_{03} \vartheta_{23} - \vartheta_{01} \vartheta_{21} \vartheta_{22}^2 \vartheta_{03} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 001 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 001 \\ 010 \end{bmatrix} &= 2c(\vartheta_{01} \vartheta_{11} \vartheta_{02} \vartheta_{12} \vartheta_{03} \vartheta_{13}) \\
\vartheta \begin{bmatrix} 001 \\ 100 \end{bmatrix} \vartheta \begin{bmatrix} 001 \\ 110 \end{bmatrix} &= 2c(\vartheta_{01} \vartheta_{11} \vartheta_{02} \vartheta_{12} \vartheta_{03} \vartheta_{13}) \\
\vartheta \begin{bmatrix} 011 \\ 110 \end{bmatrix} \vartheta \begin{bmatrix} 011 \\ 100 \end{bmatrix} &= 2c(\vartheta_{11} \vartheta_{21} \vartheta_{12} \vartheta_{22} \vartheta_{03} \vartheta_{13}) \\
\vartheta \begin{bmatrix} 101 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 101 \\ 010 \end{bmatrix} &= 2c(\vartheta_{01} \vartheta_{11} \vartheta_{12} \vartheta_{22} \vartheta_{13} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 111 \\ 000 \end{bmatrix} \vartheta \begin{bmatrix} 111 \\ 110 \end{bmatrix} &= 2c(\vartheta_{11} \vartheta_{21} \vartheta_{02} \vartheta_{12} \vartheta_{13} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 011 \\ 011 \end{bmatrix} \vartheta \begin{bmatrix} 011 \\ 111 \end{bmatrix} &= -2c(\vartheta_{11} \vartheta_{21} \vartheta_{12} \vartheta_{22} \vartheta_{03} \vartheta_{13}) \\
\vartheta \begin{bmatrix} 111 \\ 011 \end{bmatrix} \vartheta \begin{bmatrix} 111 \\ 101 \end{bmatrix} &= -2c(\vartheta_{11} \vartheta_{21} \vartheta_{02} \vartheta_{12} \vartheta_{13} \vartheta_{23}) \\
\vartheta \begin{bmatrix} 101 \\ 101 \end{bmatrix} \vartheta \begin{bmatrix} 101 \\ 111 \end{bmatrix} &= -2c(\vartheta_{01} \vartheta_{11} \vartheta_{12} \vartheta_{22} \vartheta_{13} \vartheta_{23})
\end{aligned}$$

Denote by R'_1 the product of the first four lines. Obviously $R'_1 = c(N)^4 R_1$. Calling R'_2 the product of the last fourteen lines, we get

$$R'_2 = 2^{14} \cdot c(N)^{14} \cdot \left(\prod_{i=1}^3 \vartheta_{0i}^8 \vartheta_{2i}^8 (\vartheta_{0i}^4 - \vartheta_{2i}^4)^3 \right).$$

So

$$\begin{aligned}
\chi_{18}(\tau'/2) &= R'_1 R'_2 = 2^{14} \cdot c(N)^{18} \cdot \left(\frac{2^{40}}{\pi^{54}} \cdot \det(\Omega_2)^{18} \right) \cdot X(m) \\
&= \left(\frac{2}{\pi} \right)^{54} \cdot \det(\Omega'_2)^{18} \cdot X(m),
\end{aligned}$$

which is the expected result. \square

Since $X(m)$ is equal to $T(\tilde{A})$ up to a square, Th.2.5.2 and Th.4.3.1 show Serre's conjecture.

Corollary 4.3.2. *Let $K \subset \mathbb{C}$ and $m \in S^\times$ with coefficients in K . Let $A'(m)$ be the associated abelian threefold and Ω' be one of its period matrix. Then*

$$\left(\frac{\pi}{2} \right)^{54} \cdot \chi_{18}(\Omega') \in K^{\times 2}$$

if and only if $A'(m)$ is the Jacobian of a non hyperelliptic genus 3 curve. \square

In other words, Serre's conjecture is true for our three dimensional family A of abelian threefolds.

Corollary 4.3.3. *If $m \in S^\times$ and Ω_m is a period matrix associated to the non hyperelliptic genus 3 curve X_m with Ciani form Q_m then*

$$\chi_{18}(\Omega_m) = \left(\frac{1}{2\pi} \right)^{54} \cdot \text{Disc}(Q_m)^2. \quad \square$$

Proof. Using Th.2.4.2 and (6) we get

$$\begin{aligned}
\left(\frac{\pi}{2} \right)^{54} \cdot \chi_{18}(\Omega_m) &= X(\text{Cof } m) \\
&= D(m)^2 = (2^{-54} \cdot \text{Disc } Q_m)^2.
\end{aligned}$$

\square

Remark. When $m \in S \setminus S^\times$, the abelian variety $A'(m)$ comes from a hyperelliptic curve and the above formula degenerates. However in [13] and [5] we find a beautiful formula for the hyperelliptic case in every genus. Let

$$C : Y_2 = a_{2g+2}X^{2g+2} + \dots + a_0 = a_{2g+2}(X - \alpha_1) \cdots (X - \alpha_{2g+2})$$

and

$$\Delta_{\text{alg}}(C) = a_{2g+2}^{4g+2} \prod_{j < k} (\alpha_j - \alpha_k)^2.$$

They define also a modular form on \mathbb{H}_g

$$\delta(\tau) = \prod_{\epsilon \in T} \vartheta[\epsilon](\tau)^8$$

where T is a certain subset of even theta characteristic. One has

$$\Delta_{\text{alg}}(C)^{2n} = (2\pi)^{4rg} \det(\Omega_1)^{-4r} \delta(\tau)^2$$

where

$$r = \binom{2g+2}{g+1}, \quad n = \binom{2g}{g+1},$$

and $\tau = \tau(\Omega)$ for a certain period matrix $\Omega = [\Omega_1, \Omega_2]$ of $\text{Jac}(C)$.

Remark. Denote by V_3^4 the 15-dimensional affine open set of ternary quartics. Felix Klein proved in 1889 that there is a map

$$\Omega : V_3^4 \longrightarrow \mathcal{R}_g$$

such that if $\Omega(Q) = [\Omega_1 \ \Omega_2]$ and $X : Q = 0$, then $\text{Jac } X = A_{\Omega(Q)}$ and

$$\chi_{18}(\Omega) = c \text{Disc}(Q)^2,$$

with some unspecified constant $c \in \mathbb{C}$. We prove here that $c = (1/2\pi)^{54}$. Using this precise version of Klein's formula, it is almost obvious to extend our theorem to the general case. However, we did not include it, for we think that a good presentation should include a modern proof of Klein's result. We plan to do this in a forthcoming article.

APPENDIX A.

A.1. Modularity of χ_k . Let

$$\Gamma_g(2) = \{M \in \Gamma_g \mid M \equiv \mathbf{1}_{2g} \pmod{2}\}$$

and recall that the sequence

$$1 \rightarrow \Gamma_g(2) \rightarrow \Gamma_g \rightarrow \text{Sp}_{2g}(\mathbb{F}_2) \rightarrow 1$$

is exact. We introduce the congruence subgroup

$$\Gamma_g^0(2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_g(\mathbb{Z}) \mid B \equiv 0 \pmod{2} \right\}.$$

We need a set of generators for this subgroup. For any integer $n \geq 1$, define

$$M(n) = \mathbf{M}(\mathbb{Z}) \cap \Gamma_g(n), \quad U(n) = \mathbf{U}(\mathbb{Z}) \cap \Gamma_g(n), \quad V(n) = \mathbf{V}(\mathbb{Z}) \cap \Gamma_g(n).$$

with

$$\Gamma_g(n) = \{M \in \Gamma_g \mid M \equiv \mathbf{1}_{2g} \pmod{n}\}.$$

Proposition A.1.1. *The subgroups $M(1)$, $U(2)$ and $V(1)$ generate $\Gamma_g^0(2)$, and $\Gamma_g^0(2) = \Gamma_g(2) \cdot \mathbf{M}(\mathbb{Z}) \cdot \mathbf{V}(\mathbb{Z})$.*

Proof. First, the subgroups $M(2)$, $U(2)$ and $V(2)$ generate $\Gamma_g(2)$, see [10, p. 179]. Let

$$\Gamma_g^1(2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_g(\mathbb{Z}) \mid A \equiv D \equiv 1 \pmod{2} \text{ and } B \equiv 0 \pmod{2} \right\}.$$

There is the following diagram, where the vertical arrow is the transpose of the reduction modulo 2:

$$\begin{array}{ccccccc} \Gamma_g(2) & \subset & \Gamma_g^1(2) & \subset & \Gamma_g^0(2) & \subset & \Gamma_g(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \subset & \mathbf{U}(\mathbb{F}_2) & \subset & \mathbf{P}(\mathbb{F}_2) & \subset & \mathrm{Sp}_g(\mathbb{F}_2) \end{array}$$

Then, if $M \in \Gamma_g^1(2)$ is written as usual

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{1}_g & 0 \\ C & \mathbf{1}_g \end{bmatrix} = \begin{bmatrix} A + BC & B \\ C + DC & D \end{bmatrix} \in \Gamma_g(2),$$

and if $M \in \Gamma_g^0(2)$, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & {}^tA \end{bmatrix} = \begin{bmatrix} \mathbf{1}_g & B^tA \\ CA^{-1} & D^tA \end{bmatrix} \in \Gamma_g^1(2).$$

□

Theorem A.1.2. *Assume $g \geq 3$. The function $\chi_k(\frac{1}{2}\tau)$ is a modular form on \mathbb{H}^g of weight k for $\Gamma_g^0(2)$.*

Proof. J.-I. Igusa proved [9, p. 850] that $\chi_k(\tau)$ is a modular form of weight k for $\Gamma_g(1)$ if $g \geq 3$. Let

$$H = \begin{bmatrix} \frac{1}{2}\mathbf{1}_g & 0 \\ 0 & \mathbf{1}_g \end{bmatrix}, \quad H.\tau = \frac{1}{2}\tau.$$

Let $f(\tau) = \chi_k(\frac{1}{2}\tau) = \chi_k(H.\tau)$. It is sufficient to check that

$$f(M.\tau) = j(M, \tau)^k f(\tau)$$

if M belongs to one of the generating subgroups described in Prop. A.1.1. First, if $M \in M(1)$, then $H.M = M.H$, hence,

$$f(M.\tau) = \chi_k(H.M.\tau) = \chi_k(M.H.\tau) = j(M, H.\tau)^k \chi_k(H.\tau) = j(M, \tau)^k f(\tau),$$

since $j(M, \tau) = \pm 1$ for every $M \in \mathbf{M}(\mathbb{Z})$ does not depend on $\tau \in \mathbb{H}_g$. Now, if $U \in U(2)$, then

$$U = U'^2 = \begin{bmatrix} \mathbf{1}_3 & 2B \\ 0 & \mathbf{1}_3 \end{bmatrix}, \quad \text{where } U' = \begin{bmatrix} \mathbf{1}_3 & B \\ 0 & \mathbf{1}_3 \end{bmatrix} \in \mathbf{U}(\mathbb{Z}),$$

and $H.U = H.U'^2 = U'.H$. This implies

$$f(U.\tau) = \chi_k(H.U'^2.\tau) = \chi_k(U'.H.\tau) = j(U', H.\tau)^k \chi_k(H.\tau) = j(U, \tau)^k f(\tau),$$

since $j(U^n, \tau) = 1$ for every $U \in \mathbf{U}(\mathbb{Z})$. If $V \in V(1)$, then $H.V = V^2.H$. Hence

$$\begin{aligned} f(V.\tau) &= \chi_k(H.V.\tau) = \chi_k(V^2.H.\tau) = j(V^2, H.\tau)^k \chi_k(H.\tau) \\ &= j(V, \tau)^k \chi_k(H.\tau) = j(V, \tau)^k f(\tau), \end{aligned}$$

since $j(V^2, \tau) = j(V, 2\tau)$ for every $V \in V(1)$. □

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